IPML Worksheet #1: Foundation of Imprecise Probabilities

These exercises are intended as a self-assessment of your understanding of the first part of the course. Please attempt to answer each question to the best of your ability, without consulting the lecture notes.

Lecture 1:

1. What are the **three** key ingredients of statistical machine learning?
    a. Describe each of them in detail, with particular emphasis on why they are essential to the learning problem.
    b. Provide at least one concrete example of a learning problem, and explain how each of the three ingredients manifests in this setting.

2. Describe the limitations of traditional machine learning with respect to the three key ingredients.
    a. In your opinion, which factor is the most limiting in this classical machine learning framework?
    b. How would you propose to overcome this limitation?

3. Let $\mathcal{C}$ be a concept class comprising of target concepts c. What does it mean when we say, "**a hypothesis class $\mathcal{H}$ is PAC-learnable**" with respect to this concept class?

4. What are the potential sources of uncertainty in machine learning?
    a. Describe how these sources of uncertainty influence the learning problem.

5. What are aleatoric and epistemic uncertainty?
    a. Describe the distinction between aleatoric and epistemic uncertainty.
    b. Explain how these two sources of uncertainty manifest in a machine learning problem.

6. Describe the difference between risk and uncertainty.

7. Identify one major pitfall of contemporary machine learning that arises in real-world applications.
    a. Describe how this pitfall stems from an overreliance on precise probability theory as the foundational framework of machine learning.

Lecture 2:

8. Given a finite possibility space Ω, provide an axiomatic characterization of an **additive** probability $P: \mathcal{E} \to [0,1]$ where $\mathcal{E} \subseteq 2^{\Omega}$.
    a. Based on the characterization, show that additive probability is self-conjugate, i.e., for any $A, A^c \in \mathcal{E}$, $P(A) = 1 - P(A^c)$.

9. Provide an exhaustive list of the different interpretations of probability.
    a. Describe each interpretation and provide a real-world example that illustrates the corresponding interpretation.

10. What are the concepts of avoiding sure loss and coherence?
    a. Describe the key distinction between avoiding sure loss and coherence.
    b. When do avoiding sure loss and coherence coincide?

11. Identify one major pitfall of precise probability that arises in the real-world.

12. What is an imprecise probability? Describe it in your own words.

Lecture 3:

13. Given a finite possibility space Ω, provide an axiomatic characterization of a probability interval $(\underline{p}, \overline{p})$ as a pair of lower and upper probability mass functions.
    a. Write down a credal set associated with this probability interval.

14. Consider a random variable X with a finite outcome space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. You are told that the probabilities of each elementary event are not known precisely, but lie within the following **interval constraints**:

Outcome	Lower bound l_i	Upper bound u_i
ω_1	0.2	0.4
ω_2	0.1	0.3
ω_3	0.4	0.7
ω_4	0.0	0.5

- Write the credal set $\mathcal{P} = \{p : l_i \le p(\omega_i) \le u_i, \sum_{i=1}^{4} p(\omega_i) = 1\}$.
- Compute the **lower** and **upper** probabilities of the following events:
  $$A_1 = \{\omega_1, \omega_4\},\ A_3 = \{\omega_2, \omega_3, \omega_4\},\ A_2 = \{\omega_2\}$$
- **Verify coherence:** Show that your results satisfy:
  - $\underline{P}(\emptyset) = 0,\ \overline{P}(\Omega) = 1$
  - $\underline{P}(A) + \overline{P}(\bar{A}) = 1$
  - $\underline{P}(A) \le \overline{P}(A)$ for all A

- **Interpret the Results:**
    - What is the meaning of the gap $\overline{P}(A) - \underline{P}(A)$?
    - What kind of uncertainty does this gap reflect?
    - For which event do you observe the highest imprecision, and why?

15. Given a gamble f on a finite possibility space Ω, i.e., a bounded real-valued function on Ω, define lower prevision $\underline{P}(f)$ and upper prevision $\overline{P}(f)$ in terms of the gamble f.
    a. How would you interpret the lower and upper previsions from the betting perspective?
    b. Show that $\overline{P}(f) = -\underline{P}(-f)$, i.e., $\overline{P}$ is a conjugate of $\underline{P}$.
    c. Why is $\underline{P}$ not self-conjugate, unlike the standard precise probability P? What additional information does this lack of self-conjugacy capture?

16. Describe the Principle of Minimal Specificity with specific real-world examples.

17. Given a possibility distribution $\pi: \Omega \to [0,1]$, provide an axiomatic characterization of possibility measure Π and necessity measure N.
    a. Show that possibility and necessity measures correspond to upper and lower probability, respectively.
    b. For an event A, how would you interpret $\Pi(A)$ and $N(A)$?
    c. Show the possibility-necessity duality: $\Pi(A) = 1 - N(A^c)$.

18. Describe the qualitative and quantitative interpretations of possibility theory.

19. Identify one application of possibility theory in machine learning.
    a. Explain why possibility theory is the appropriate tool for addressing this problem.

Lecture 4:

20. Given a frame of discernment $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, consider the following mass function:

$$m(\{\omega_1, \omega_2\}) = 0.40, \;\; m(\{\omega_1, \omega_3\}) = 0.35, \;\; m(\{\omega_4\}) = 0.25$$

    a. List all the focal sets of m.
    b. Find the core of m.

21. Given a frame of discernment $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, specify the following mass functions:
    a. A logical mass function of $A = \{\omega_1, \omega_2, \omega_3\}$.
    b. A vacuous mass function that represents total ignorance.
    c. A Bayesian mass function that is equivalent to a probability distribution

22. Consider a frame of discernment $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with mass function $m(\{\omega_2\}) = {}^1\!/_4$ and $m(\Omega) = {}^3\!/_4$.

    a. Compute the belief function Bel induced by m.
    b. Compute the plausibility function Pl induced by m.
    c. For the event $A = \{\omega_2, \omega_3\}$, compute the values of $Bel(A)$ and $Pl(A)$.

23. Consider the following scenario. A pandemic has caused widespread illness in a city. As a scientist, your task is to investigate its underlying cause. You hypothesise that the pandemic is caused by one of three possible agents: Virus A, Virus B, or Virus C. As part of your investigation, you receive information from two sources: an experimental study and an expert opinion.

Ω_1: Experiment		**Ω_2: Expert Opinion**	
Unreliable ($p = 0.2$)	Reliable ($p = 0.8$)	Experienced ($p = 0.6$)	Inexperienced ($p = 0.4$)
$\{A, B, C\}$	$\{A, B\}$	$\{B, C\}$	$\{A, B, C\}$

When the experimental outcome is unreliable, the only conclusion that can be drawn is that the true cause is one of Virus A, Virus B, or Virus C. When the experiment is reliable, the conclusion can be narrowed to either Virus A or Virus B.

In contrast, the conclusion drawn from an expert opinion depends on the expert's level of experience. If the expert is experienced, one may conclude that the true cause is either Virus B or Virus C. If the expert is inexperienced, however, no such refinement is possible, and the only conclusion is again that the true cause is one of Virus A, Virus B, or Virus C.

a. Identify the mass functions m_1 and m_2 associated with these two sources of information.
b. Calculate the degree of conflict between m_1 and m_2.
c. Apply the Dempster's rule of combination to obtain $m_1 \oplus m_2$.
d. Calculate the belief and plausibility functions induced by the mass function $m_1 \oplus m_2$.